

# Robustify and Tighten the Lee Bounds: A Sample Selection Model under Stochastic Monotonicity and Symmetry Assumptions

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**Summary** In the presence of sample selection, Lee’s (2009) nonparametric bounds are a popular tool for estimating a treatment effect. However, the Lee bounds rely on the monotonicity assumption, whose empirical validity is sometimes unclear. Furthermore, the bounds are often regarded to be wide and less informative even under monotonicity. To address these issues, this study introduces a stochastic version of the monotonicity assumption alongside a nonparametric distributional shape constraint. The former enhances the robustness of the Lee bounds with respect to monotonicity, while the latter helps tighten these bounds. The obtained bounds do not rely on the exclusion restriction and can be root- $n$  consistently estimable, making them practically viable. The potential usefulness of the proposed methods is illustrated by their application to experimental data from the after-school instruction programme studied by Muralidharan, Singh, and Ganimian (2019).

**Keywords:** *attrition, partial identification, randomised controlled trial, treatment effects*

## 1. INTRODUCTION

Sample selection is an important challenge for identifying causal effects in empirical economics (see, e.g., Hansen, 2022, p. 881). Earlier approaches<sup>1</sup> model the selection mechanism as follows:

$$Y_i^* = X_i' \beta + \varepsilon_i, \quad Y_i = Y_i^* \mathbf{1}\{W_i' \gamma + \nu_i \geq 0\},$$

where  $Y_i^*$  is the latent outcome and researchers observe  $(Y_i, X_i, W_i)$ ; and assume that the covariate  $W_i$  includes some variables which are not in  $X_i$ , meaning that researchers can access some “instrument” that only affects an individual’s selection decision but not the outcome. A similar assumption is made by not only such single index semiparametric models but more general nonparametric models (e.g., Das et al., 2003). However, this exclusion restriction is not always satisfied in practice because such an instrument may not be easily accessible. To overcome this challenge, Lee (2009) develops a partial identification strategy that does not necessitate this restriction. The Lee bounds are especially used in field experimental studies, wherein the attrition problem often occurs (e.g., Baranov et al., 2020; Bursztyn et al., 2020; Muralidharan et al., 2019). Another application includes Chen and Roth (2023).

Although popular, the Lee bounds have two important limitations. The first is the monotonicity assumption. The identification of the Lee bounds relies on the assumption of a monotone response to the treatment assignment; specifically, *for every person*, their outcome under treatment has to be observable if it would be observed in the absence

<sup>1</sup>E.g., Heckman (1979). See Heckman (1990) and references therein for related specifications.

of treatment. This assumption, while classic and seemingly natural in many contexts, is somewhat strong and its empirical validity can be unclear in several cases.

The second limitation is that the Lee bounds can be wide. For example, Mobarak et al. (2023, p. 383) wrote, “[c]ommon nonparametric bounds (e.g., Lee 2009) are often wide and uninformative, as they trim the data from the extremes of the outcome distribution.” Several studies provide a similar assessment (e.g., Barrow and Rouse, 2018; Delius and Sterck, 2024). Therefore, considering the Lee bounds are the tightest possible under his assumptions, it will be beneficial to make an additional, reasonable restriction to help draw more informative policy implications.

This article aims to address these issues by introducing new choices of assumptions. We first relax the monotonicity assumption to its stochastic counterpart, which allows a certain stochastic deviation from the monotone response. This *stochastic monotonicity assumption* operates on the intuition that “the monotonicity approximately, but perhaps not exactly, holds.” Such scenarios will frequently arise in empirical analysis. Moreover, embracing a stochastic variation to choice aligns more closely with observed experimental evidence (e.g., Tversky, 1969; Agranov and Ortoleva, 2017). Afterwards, we introduce a nonparametric distributional assumption that helps tighten the bounds. Specifically, we assume that the density function of the latent outcome  $Y_i^*$  under treatment for always-takers is (approximately) symmetric. This *symmetry assumption* restricts the admissible class of trimmed densities, enabling us to avoid Lee’s (2009) “extreme” trimming of the outcome distribution, thereby tightening the bounds over the standard Lee bounds.

Based on these assumptions, we develop partial identifications and propose simple estimators. Importantly, the bounds are root- $n$  consistently estimable, making the implementation practically feasible. The potential usefulness of the bounds is demonstrated using an empirical example.

This article proceeds as follows. The rest of this section reviews the related literature. In Section 2, we discuss the identification of the sample selection model under the stochastic monotonicity assumption. Section 3 introduces the symmetry assumption and establishes identification. Some numerical examples are also provided in these sections. Estimation and inference procedures are considered in Section 4. The empirical illustration in Section 5 highlights the usefulness of the obtained bounds. Section 6 concludes this article.

### Related Literature

The sample selection problem has been an important issue in economics (Heckman, 1979). Lee (2009) is a seminal work that derives nonparametric bounds under the monotonicity assumption. Similar studies include Imai (2008) and Chen and Flores (2015). Based on a similar idea, Bartalotti et al. (2023) recently considers the identification of the marginal treatment effect in the presence of sample selection, which essentially extends Lee’s model.

This study is not the first to focus on the monotonicity assumption. Relatedly, Semenova (2023) explores a different kind of generalisation of the Lee bounds, specifically identifying the bounds assuming that exact monotonicity holds conditional on the covariates. This approach will be very useful when the direction of selection systematically differs depending on the covariates. However, in some cases, researchers may not have access to such informative covariates. Besides, even conditional on the covariates, the requirement that the monotone response in selection has to hold for every such person may still be practically strong. Hence, when the predictive power of the covariates seems

weak but one can assume that the fraction of those deviating from complete monotonicity is not so large, our approach can be a useful complementary tool.

Our relaxation of the exact monotonicity assumption uses the information of the lower bound on a parameter that represents how much the monotonicity is plausible. This is similar to the techniques used in the sensitivity analysis literature, such as Conley et al. (2012), which treats the exogeneity condition in linear instrumental variable methods. Such an idea has also been applied to treat non-classical measurement error problems (e.g., Horowitz and Manski, 1995; Kreider et al., 2012). The stochastic monotonicity assumption is also similar to the bounded variation assumption found in the difference-in-differences and regression discontinuity literature (e.g., Imbens and Wager, 2019; Manski and Pepper, 2018; Rambachan and Roth, 2023).

The width of the Lee bounds is also discussed in the literature. Behaghel et al. (2015) proposes an interesting idea to derive narrower nonparametric bounds on the treatment effect, whereas an instrument that only affects the participation decision is needed. Honoré and Hu (2020, 2024) derives much tighter bounds than Lee bounds under fully structured semiparametric models. Although their results do not rely on the excluded instrument, their identification depends on the imposed structural assumptions, meaning that it is less robust to specification error than nonparametric models as Lee's (2009). Intuitively, our approach utilising (nonparametric) distributional assumption is inbetween these two studies: Our procedure does not require the additional instruments but still builds on Lee's insight.

Finally, our procedure can be applicable to various empirical settings where sample selection can occur. A prominent example is the attrition problem in randomised controlled trials (Duflo et al., 2007, Section 6.4). To demonstrate such an application, we use the data from Muralidharan et al.'s (2019) study as an empirical example in Section 5.

## 2. BOUNDS UNDER STOCHASTIC MONOTONICITY

Here, we introduce a stochastic version of the monotonicity assumption and provide partial identification under this assumption. Section 2.1 reviews the general selection model of Lee (2009) and the monotonicity assumption. In Section 2.2, we extend Lee's (2009) results to establish the more robust bounds under the stochastic monotonicity.

### 2.1. General Selection Model and Monotonicity Assumption

Following Lee (2009), we consider the following general selection model:

$$(Y_{1i}^*, Y_{0i}^*, S_{1i}, S_{0i}, D_i) \text{ is i.i.d. across individuals,} \quad (2.1a)$$

$$S_i = S_{1i}D_i + S_{0i}(1 - D_i), \quad (2.1b)$$

$$Y_i = S_i \cdot \{Y_{1i}^*D_i + Y_{0i}^*(1 - D_i)\}, \quad (2.1c)$$

$$(Y_i, S_i, D_i) \text{ is observed,} \quad (2.1d)$$

where  $Y_{1i}^*$  ( $Y_{0i}^*$ ) is the potential outcome when an individual  $i$  is assigned to a treatment (control) group;  $D_i \in \{0, 1\}$  is a binary indicator of treatment status that equals 1 when the individual is treated; and  $S_{1i} \in \{0, 1\}$  and  $S_{0i} \in \{0, 1\}$  are the potential sample

selection indicator under the treated and controlled states, respectively.  $S_i$  equals 1 when the individual  $i$  does not exhibit attrition, and 0 otherwise.<sup>2</sup>

We are interested in the average treatment effect for the always takers ( $S_{0i} = S_{1i} = 1$ ), or

$$\tau := \mathbb{E}[Y_{1i}^* - Y_{0i}^* | S_{0i} = S_{1i} = 1].$$

Lee (2009) shows that  $\tau$  can be partially identified under the following assumptions.<sup>3</sup>

ASSUMPTION 2.1. (INDEPENDENT ASSIGNMENT)  $(Y_{1i}^*, Y_{0i}^*, S_{1i}, S_{0i})$  is independent of  $D_i$ .

ASSUMPTION 2.2. (MONOTONICITY)  $\mathbb{P}[S_{1i} = 1 | S_{0i} = 1] = 1$ .

The basic idea of the Lee bounds is as follows. Under Assumptions 2.1 and 2.2, we have  $\tau = \mathbb{E}[Y_{1i}^* | S_{0i} = S_{1i} = 1] - \mathbb{E}[Y_i | D_i = 0, S_i = 1]$ . The point identification fails due to the first term of the right-hand side, as it is not identified. Here, under Assumption 2.2, we have the following decomposition:

$$\begin{aligned} & \underbrace{\mathbb{P}[Y_i \leq y | D_i = 1, S_i = 1]}_{\text{identified}} \\ &= \mathbb{P}[Y_{1i}^* \leq y | S_{0i} = 0, S_{1i} = 1] (1 - q_0) + \underbrace{\mathbb{P}[Y_{1i}^* \leq y | S_{0i} = 1, S_{1i} = 1]}_{\text{our interest}} q_0, \end{aligned} \quad (2.2)$$

where  $q_0 = \mathbb{P}[S_i = 1 | D_i = 0] / \mathbb{P}[S_i = 1 | D_i = 1]$ , which is identified and in  $[0, 1]$  under Assumption 2.2. By assuming that the smallest  $q_0$  values of  $Y_i$  are entirely attributed to the group with  $S_{0i} = 1, S_{1i} = 1$ , we can derive the lower bound as follows:

$$\mathbb{E}[Y_{1i}^* | S_{0i} = S_{1i} = 1] \geq \underbrace{\mathbb{E}[Y_i | D_i = 1, S_i = 1, Y_i \leq y_{q_0}^1]}_{\text{identified}},$$

where  $y_{q_0}^1$  is the  $q_0$ th quantile of  $Y_i | D_i = 1, S_i = 1$ . This is Lee's lower bound. The upper bound can be obtained similarly.

Although it plays several important roles in the above transformation, the monotonicity assumption is somewhat restrictive. Intuitively speaking, it requires that *every individual* in the population obeys the behavioural rule that  $S_{0i} = 1 \implies S_{1i} = 1$  *with no exception*. This may fail if the population consists of heterogeneous individuals. Further, as discussed in Section 5 using an empirical example, several context-specific concerns regarding its plausibility may exist.

However, if the monotonicity assumption is roughly true, would the Lee bounds successfully produce valid bounds? Unfortunately, a slight deviation from the exact monotonicity can lead to misleading conclusions, as illustrated in the next example:

<sup>2</sup>We drop covariates throughout this article, but researchers can use baseline covariates to obtain possibly narrower bounds in the same manner as Lee (2009, p. 1086). See the Online Appendix (Section S1.3) for a detailed discussion.

<sup>3</sup>This  $\tau$  may not necessarily be of primary interest, as criticised in Honoré and Hu (2024). However, it is the only aspect that can be learnt under Lee's or our assumptions, and additional restrictions are needed to obtain further information. Still,  $\tau$  is likely to be one of the important parameters in many contexts; see also Lee (2009, pp. 1085–1086) for further discussion.

EXAMPLE 2.1.  $D_i$  is assigned randomly. On the selection mechanism, suppose

$$\begin{cases} \mathbb{P}[S_{1i} = 1, S_{0i} = 1] = 0.475, & \mathbb{P}[S_{1i} = 0, S_{0i} = 1] = 0.025, \\ \mathbb{P}[S_{1i} = 1, S_{0i} = 0] = 0.300, & \mathbb{P}[S_{1i} = 0, S_{0i} = 0] = 0.200. \end{cases}$$

Thus, a small fraction (2.5%) of the population deviates from the monotone response. On the outcome, suppose that the conditional densities are given by

$$\begin{cases} f_{Y_0^*}(y|S_{1i} = k, S_{0i} = 1) = \phi(y; 0, 1/2), & f_{Y_1^*}(y|S_{1i} = k, S_{0i} = 1) = \phi(y; 0, 1/2), \\ f_{Y_0^*}(y|S_{1i} = 1, S_{0i} = 0) = \phi(y; 2, 1/2), & f_{Y_1^*}(y|S_{1i} = 1, S_{0i} = 0) = \phi(y; 2, 1/2), \end{cases}$$

where  $k \in \{0, 1\}$  and  $\phi(\cdot; \mu, \sigma^2)$  is the density function of the normal distribution  $\mathcal{N}(\mu, \sigma^2)$ . The treatment has no effects for always takers, i.e.,  $\tau = 0$ . The Lee bounds under the monotonicity assumption are computed as  $[0.02, 1.46]$ , which do not cover the true  $\tau$ .

Therefore, when the monotonicity does not exactly hold, the Lee bounds can be invalid.

## 2.2. Stochastic Monotonicity and Partial Identification

When computing the Lee bounds, economists often operate under the premise that the monotonicity assumption is largely, though perhaps not perfectly, met. As highlighted by the above example, considering bounds that account for this nuanced understanding can enhance robustness. To align with this perspective, this subsection introduces a weaker and interpretable assumption that builds on the intuition that “the monotonicity mostly, but perhaps not exactly, holds” and then derives the bounds on the treatment effect  $\tau$  under this assumption.

In particular, we consider the identification under the following stochastic version of the monotonicity:

ASSUMPTION 2.3. (STOCHASTIC MONOTONICITY)

- (i)  $\mathbb{P}[S_{1i} = 1|S_{0i} = 1] \geq \vartheta_L$  with known  $\vartheta_L \in (0, 1]$ .
- (ii)  $(Y_{1i}^*, Y_{0i}^*)$  is independent of  $S_{1i}$  conditional on  $S_{0i} = 1$ .

REMARK 2.1. Assumption 2.2 is a special case of Assumption 2.3 with  $\vartheta_L = 1$ .

Part (i) states that the monotonicity holds with a probability of no less than some known value  $\vartheta_L$ . This  $\vartheta_L$  can be considered a representation of the validity of the monotonicity. It can be determined based on a researcher’s institutional knowledge and beliefs. For example, when a researcher assumes that almost all participants satisfy the monotonicity assumption but anticipates that a small fraction of them perhaps deviate from the monotonic behaviour, setting  $\vartheta_L = 0.95$  could be reasonable. This kind of bound on a parameter is common in various settings (e.g, Conley et al., 2012; Horowitz and Manski, 1995; Kreider et al., 2012; Imbens and Wager, 2019; Rambachan and Roth, 2023), whereas its intuitive meaning is not always clear. In our setting,  $\vartheta_L$  has a clear interpretation as it represents a bound on probability, making it easier to align with the researcher’s knowledge. For instance, an assumption like “largely reasonable” might correspond to 0.95, and “half of them could satisfy this” to 0.5, which is more straightforward to map than, for example, specifying bounds on the second derivative of a regression function (Imbens and Wager, 2019).

The lower bound  $\vartheta_L$  provides another intuitive relation. Under Assumption 2.1, we have

$$\mathbb{P}[S_{1i} = 1|S_{0i} = 0] = \frac{\mathbb{P}[S_i = 1|D_i = 1] - \mathbb{P}[S_{1i} = 1|S_{0i} = 1]\mathbb{P}[S_i = 1|D_i = 0]}{1 - \mathbb{P}[S_i = 1|D_i = 0]}. \quad (2.3)$$

Hence, under Assumption 2.3-(i), it follows that

$$\mathbb{P}[S_{1i} = 1|S_{0i} = 0] \leq \frac{\mathbb{P}[S_i = 1|D_i = 1] - \vartheta_L \mathbb{P}[S_i = 1|D_i = 0]}{1 - \mathbb{P}[S_i = 1|D_i = 0]}. \quad (2.4)$$

That is, setting  $\vartheta_L$  is equivalent to determining the upper bound on  $\mathbb{P}[S_{1i} = 1|S_{0i} = 0]$  once the identified parameters are known. This could be used to empirically validate Assumption 2.3-(i) by computing the empirical analogue of the upper bound and checking if  $\mathbb{P}[S_{1i} = 1|S_{0i} = 0]$  is not too small in light of the researcher's beliefs.

Part (ii) states that the attrition and potential outcomes are irrelevant for those whose outcomes are (potentially) observable without treatment. Lee's monotonicity assumption automatically satisfies this. However, it is not explicitly treated in Lee (2009), and thus, we discuss this assumption a little more. Assumption 2.3-(ii) can be rewritten as

$$\mathbb{P}[S_{1i} = 1|Y_{1i}^*, Y_{0i}^*, S_{0i} = 1] = \mathbb{P}[S_{1i} = 1|S_{0i} = 1]. \quad (2.5)$$

Importantly,  $Y_{1i}^*, Y_{0i}^*$  can affect  $S_{1i}$  only through  $S_{0i}$  if  $S_{0i} = 1$ . When we suppose that  $S_{1i}$  still depends on  $Y_{1i}^*, Y_{0i}^*$  after controlling  $S_{0i} = 1$ , condition (2.5) will fail. In contrast, this condition can be satisfied, for example, when

$$S_{0i} = \mathbf{1}\{k_0(Y_{1i}^*, Y_{0i}^*) \geq \zeta_{0i}\}, S_{1i} = \mathbf{1}\{k_1(S_{0i}) \geq \zeta_{1i}\} \text{ for those } S_{0i} = 1, \quad (2.6)$$

where  $k_0$  and  $k_1$  are some structural functions, and  $\zeta_{0i}$  and  $\zeta_{1i}$  are idiosyncratic errors. This behavioural structure includes the following examples:

**EXAMPLE 2.2.** Consider an off-the-job training programme for employees and its effect on earnings.  $D_i$  is the treatment indicator of the job-training, and  $Y_{1i}^*, Y_{0i}^*$  are earnings.  $S_{0i}, S_{1i}$  denote labour force participation. Assume that the decision-makers participate in the labour market only if their subjectively expected earnings are above the predetermined reservation wage  $r_i (> 0)$ ;  $Y_{di}^*$  is decomposed as the sum of the deterministic baseline ( $b_i$ ) and stochastic performance-based pay ( $p_{di}$ ), i.e.,  $Y_{di}^* = b_i + p_{di}$ ; and  $\mathbb{E}_i^{\text{sbj}}[p_{1i}] = \mathbb{E}_i^{\text{sbj}}[p_{0i}] + e_i$ , where  $e_i \geq 0$  and  $\mathbb{E}_i^{\text{sbj}}$  denotes the subjective expectation. This may be likely when the potential benefit is explained by the programme staff. In this setting, for those  $S_{0i} = 1$ , it follows that

$$S_{0i} = \mathbf{1}\{\mathbb{E}_i^{\text{sbj}}[Y_{0i}^*] \geq r_i\} = \mathbf{1}\{b_i + \mathbb{E}_i^{\text{sbj}}[p_{0i}] \geq r_i\},$$

$$S_{1i} = \mathbf{1}\{\mathbb{E}_i^{\text{sbj}}[Y_{1i}^*] \geq r_i\} = \mathbf{1}\{b_i + \mathbb{E}_i^{\text{sbj}}[p_{0i}] + e_i \geq r_i\} = \mathbf{1}\{S_{0i} = 1\} \text{ for those } S_{0i} = 1,$$

This is a special case of (2.6). Hence, Assumption 2.3-(ii) is satisfied. Assumption 2.2 also holds.

**EXAMPLE 2.3.** Consider again the situation in Example 2.2. Assume here that the job training requires some utility cost  $c_i \in \{0, \infty\}$  that is independent of  $(Y_{1i}^*, Y_{0i}^*)$ ,  $0 < \mathbb{P}[c_i = \infty] \ll 1$ . The setup is a simplification made to indicate that, for the most part, the utility cost is sufficiently small. Now, suppose that

$$S_{0i} = \mathbf{1}\{\mathbb{E}_i^{\text{sbj}}[Y_{0i}^*] \geq r_i\}, S_{1i} = \mathbf{1}\{\mathbb{E}_i^{\text{sbj}}[Y_{1i}^*] - c_i \geq r_i\}.$$

This may happen when the job training programme is held at night after clocking out and can be physically/mentally very hard for someone;  $c_i = \infty$  indicates that the programme is so hard for individual  $i$  that, say, goes on sick leave. Then,

$$S_{0i} = \mathbf{1}\{b_i + \mathbb{E}_i^{\text{sbj}}[p_{0i}] \geq r_i\},$$

$$S_{1i} = \mathbf{1}\{b_i + \mathbb{E}_i^{\text{sbj}}[p_{0i}] + e_i - c_i \geq r_i\} = \mathbf{1}\{S_{0i} \times \mathbf{1}\{c_i = 0\} = 1\} \text{ for those } S_{0i} = 1,$$

which is a special case of (2.6). In this case, Assumption 2.3-(ii) holds, but Assumption 2.2 fails. If we can assume  $\mathbb{P}[c_i < \infty] \geq \vartheta_L$  with some  $\vartheta_L$ , Assumption 2.3-(i) holds.

As illustrated in Example 2.3, Assumption 2.3 accounts for scenarios where Assumption 2.2 is largely reasonable, yet subject to potential deviations due to *stochastic shocks* affecting the decision-makers' tastes. Note that, other than this rather restrictive model's setup in Example 2.3, such a stochastic flip of choice is widely acknowledged in decision theory, as evidenced by Tversky (1969) and Agranov and Ortoleva (2017).

Under this stochastic monotonicity assumption, we have the following bounds.

**THEOREM 2.1.** *Suppose the model (2.1). Let  $Y_{0i}^*$  and  $Y_{1i}^*$  be continuous random variables. Under Assumptions 2.1 and 2.3, the sharp bounds  $[\Delta, \nabla]$  are given by*

$$\nabla = \mathbb{E}[Y_i | D_i = 1, S_i = 1, Y_i \geq y_{1-\vartheta q_0}^1] - \mathbb{E}[Y_i | D_i = 0, S_i = 1],$$

$$\Delta = \mathbb{E}[Y_i | D_i = 1, S_i = 1, Y_i \leq y_{\vartheta q_0}^1] - \mathbb{E}[Y_i | D_i = 0, S_i = 1],$$

where  $y_r^1 = \inf\{t \in \mathbb{R} : r \leq F_1(t)\}$  with  $F_1$  the cumulative distribution function of  $Y_i$  conditional on  $D_i = 1, S_i = 1$ ,  $q_0 = \mathbb{P}[S_i = 1 | D_i = 0] / \mathbb{P}[S_i = 1 | D_i = 1]$ ,  $\vartheta = \max\{\vartheta_L, \vartheta_F\}$ ,  $\vartheta_F = 1 + 1/q_0 - 1/\alpha_0$ , and  $\alpha_0 = \mathbb{P}[S_i = 1 | D_i = 0]$ .

The max operator and  $\vartheta_F$  are due to the Fréchet inequality, which is equivalent to the inequality constraint  $\mathbb{P}[S_{1i} = 1 | S_{0i} = 0] \leq 1$ ; see also (2.3). This ensures that  $\mathbb{P}[S_{1i} = 1 | S_{0i} = 1] \geq \vartheta_F$  under Assumption 2.1. Therefore, the trimming is not performed at  $y_{\vartheta q_0}^1$  but at  $y_{\vartheta q_0}^1$ .

When  $\vartheta_L = 1$ , the bounds coincide with Lee's (2009). As can be seen from Theorem 2.1, the length between the bounds becomes wider as  $\vartheta_L$  decreases as long as  $\vartheta_L \geq \vartheta_F$ . This is natural since the less information we have about the individual's selection rule, the greater the length of the bounds. With this monotonic relation between  $\vartheta_L$  and the bounds, the bounds under Assumption 2.3 can also be used to perform sensitivity analysis or examine the monotonicity assumption's identifying power. For example, when researchers are concerned about whether the monotone response assumption is reasonable, drawing the bounds with multiple  $\vartheta_L$  will be helpful. If the bounds do not cover zero with somewhat smaller  $\vartheta_L$ , researchers can safely conclude that the treatment has a positive effect.

In the previous subsection, we saw a simple example where a small deviation from the monotonicity leads to misleading bounds. The stochastic monotonicity can produce more robust bounds:

**EXAMPLE 2.4. (EXAMPLE 2.1 CONTINUED)** *Assume the stochastic monotonicity assumption holds with  $\vartheta_L = 0.95$ . Then,  $\vartheta = 0.95$  as  $\vartheta_F = 0.55$ . In contrast to the bounds obtained in Example 2.1, the bounds under stochastic monotonicity are  $[-0.04, 1.53]$ , which correctly covers zero (true effect).*

While the bounds have been made more robust with respect to the deviation from monotonicity, this robustification may lead to wider bounds that could obscure policy implications. This is especially concerning given that the standard Lee bounds are already considered to be wide. The subsequent section explores a method to tighten these bounds while maintaining robustness against deviations from monotonicity.

### 3. TIGHTENING BOUNDS USING A SYMMETRY ASSUMPTION

#### 3.1. Motivation and Idea

The Lee bounds are often argued to be too wide and less informative even under the exact monotonicity assumption (e.g., Barrow and Rouse, 2018; Delius and Sterck, 2024; Mobarak et al., 2023). To address this practical issue, we introduce an additional assumption to tighten the bounds. As this assumption does not require additional variables that satisfy the exclusion restriction, the following identification result can be useful in the absence of such excluded variables. With such an additional variable at hand, the procedure proposed in Behaghel et al. (2015) may be an alternative tool.

Before introducing the assumption, we again recall the idea behind the Lee bounds. This clarifies why the bounds can be wide and motivates our assumption as a natural extension. For ease of exposition, temporarily assume that the exact monotonicity holds. As noted in Section 2.1, the point identification fails due to  $\mathbb{E}[Y_{1i}^* | S_{0i} = S_{1i} = 1]$ . An important insight from Lee (2009) was the decomposition (2.2), and we obtain Lee’s lower bound as

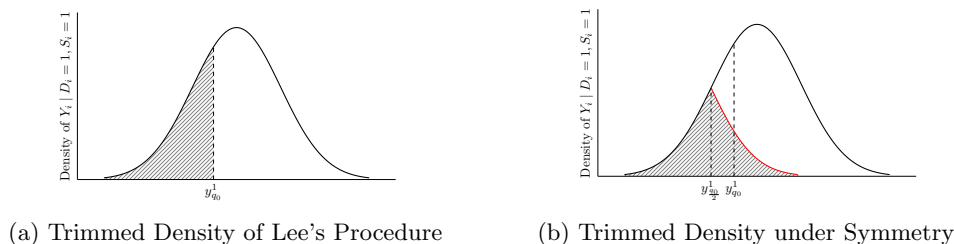
$$\mathbb{E}[Y_{1i}^* | S_{0i} = S_{1i} = 1] \geq \mathbb{E}[Y_i | D_i = 1, S_i = 1, Y_i \leq y_{q_0}^1],$$

by assuming that the smallest  $q_0$  values of  $Y_i$  are entirely attributed to  $Y_{1i}^*$  of the group with  $S_{0i} = S_{1i} = 1$ . Importantly, this is equivalent to assuming that the conditional density of  $Y_{1i}^*$  given  $S_{0i} = S_{1i} = 1$  can, after appropriate rescaling, have the form like the shaded area in Figure 1a, wherein the solid line represents the density of  $Y_i$  conditional on  $D_i = 1, S_i = 1$ .

Is this form, right-truncated at  $y_{q_0}^1$ , empirically natural? In several cases, this may be counter-intuitive and perhaps makes practitioners suppose that “Lee Bounds are based on extreme assumptions about sample selection” (Delius and Sterck, 2024).

In many empirical contexts, a “well-behaved” density would be more probable. One promising candidate is the class of symmetric densities, especially because the distribution of several economically important variables frequently shows approximate symmetry. For example, log-consumption (Battistin et al., 2009), log-wages (Online Appendix), test scores (Estevan et al., 2018, Figure 1), health outcomes such as birthweight (Maruyama and Heinesen, 2020, Figure 1) and height (A’Hearn et al., 2009, Figure 3), and productivity (Adhvaryu et al., 2022, Figure 1) are found to exhibit the Gaussian or similar mostly symmetric distributions. Hence, assuming the symmetry would be a good additional layer of assumptions (Manski and Nagin, 1998) for such contexts.

Assuming the conditional density is symmetric, we can avoid the aforementioned right-truncated trimmed density, and the lower bound is attained when it has the form as in Figure 1b, wherein the shaded area is constructed by folding the solid line to the left of  $y_{q_0/2}^1$  toward the right. Such a functional form may not be counter-intuitive and perhaps more natural in practice. Additionally, this additional information could tighten the

**Figure 1:** Comparison of Trimmed Densities

bounds. Further, the symmetry immediately implies that the lower bound is given by

$$\mathbb{E}[Y_{1i}^* | S_{0i} = S_{1i} = 1] \geq y_{\frac{q_0}{2}}^1,$$

which can be easily estimated, bypassing the need for, for example, a first-stage non-parametric estimation. The following subsection formalises the symmetry assumption and provides identification.

### 3.2. Assumptions and Partial Identification

The previous subsection introduced the symmetric density assumption in a somewhat ad-hoc manner. When, then, is this assumption plausible in practice? To investigate this, note that we can express  $Y_{1i}^* = Y_{0i}^* + \tau + \epsilon_i$ , where  $\epsilon_i$  captures the variation in the individual treatment effects and  $\mathbb{E}[\epsilon_i | S_{0i} = S_{1i} = 1] = 0$ . It then becomes clear that the validity of the symmetry assumption hinges on the (conditional joint) distribution of  $Y_{0i}^*$  and  $\epsilon_i$ . This observation indicates that the symmetry holds if the following three conditions are satisfied: conditional on  $S_{0i} = S_{1i} = 1$ , (i) the distribution of  $Y_{0i}^*$  is symmetric, (ii)  $\epsilon_i$  does not depend on  $Y_{0i}^*$ , and (iii)  $\epsilon_i$  is symmetrically distributed. It would be useful in practice to consider these points in the manner outlined below.

The first condition can be assessed by referencing the existing empirical observations and will be reasonable if the outcome of interest often exhibits a symmetric density, as exemplified in the previous subsection. For example, it is natural to assume that log wages without any treatment follow a Gaussian-like distribution, as commonly assumed and also observed in the Online Appendix. The second condition can be evaluated based on the nature of the treatment. For example, if the job training offers a lecture on documentation skills and the treatment is (randomly) provided to similar employees, such as a group of new hires, the log-wages ( $Y_{0i}^*$ ) and  $\epsilon_i$  may not be correlated, thereby satisfying the second condition. This kind of justification of the second condition is further discussed in Section 5 using a concrete empirical example. Finally, the third condition becomes plausible if (ii) holds. In this case,  $\epsilon_i$  can be viewed as an “error” term, which can reasonably be assumed to be symmetric, much like Gaussian errors.

We recommend that a researcher first check these points. If all three are reasonable, the symmetry would be justified. However, it may be challenging for researchers to justify condition (ii) in some cases. For instance, if the lecture on documentation skills is provided to a diverse group of employees, ranging from new hires to managers, a negative correlation between  $Y_{0i}^*$  and  $\epsilon_i$  could arise because less-experienced workers would benefit more (i.e., smaller  $Y_{0i}^*$  leads to larger  $\epsilon_i$ ). In such a case, it would be useful to assess whether the two variables do not exhibit a nonlinear relationship. For example,

symmetry can still hold when  $(Y_{0i}^*, \epsilon_i)$  follows the bivariate normal distribution with arbitrary correlation. Similarly, if  $\epsilon_i = \phi(Y_{0i}^*) + \zeta_i$ , where  $\phi$  is an affine function,  $Y_{0i}^* \perp \zeta_i$ , and  $\zeta_i$  is a symmetric error, the conditional density remains symmetric as long as (i) is met. That is, if the relationship between  $Y_{0i}^*$  and  $\epsilon_i$  is expected to be nearly linear, the symmetry assumption is likely reasonable. In the aforementioned example, this would likely be the case. On the other hand, if a strongly nonlinear relationship is anticipated, it may be better to use the results from Theorem 2.1. For instance, if the lecture consists of two sessions—documentation skills and management skills—then new hires (low  $Y_{0i}^*$ ) and managers (high  $Y_{0i}^*$ ) benefit from the lecture, while staff (intermediate  $Y_{0i}^*$ ) may not. In such a scenario, the relationship could be nonlinear, possibly leading to a skewed conditional distribution of  $Y_{1i}^*$  even when (i) holds.

Now, we formally introduce a version of the symmetry assumption that is slightly weaker than exact symmetry and can thus be applied when the discussion above is reasonable in an approximate sense:

ASSUMPTION 3.1. (MEAN-MEDIAN COINCIDENCE)

$$\text{Median}[Y_{1i}^* | S_{0i} = S_{1i} = 1] = \mathbb{E}[Y_{1i}^* | S_{0i} = S_{1i} = 1].$$

Next, we introduce the following tail smoothness and positivity conditions, which are implicitly used in Figure 1b. These assumptions are not essential to obtain valid bounds.

ASSUMPTION 3.2. (TAIL SMOOTHNESS & POSITIVITY) *Let  $f_y^1$  be the density function of  $Y_i$  conditional on  $D_i = 1$  and  $S_i = 1$ . Given  $\vartheta$  and  $q_0$  (defined in Theorem 2.1), (i) for any  $a \in \mathbb{R}_+$ ,*

$$f_y^1\left(y_{\frac{\vartheta q_0}{2}}^1 - a\right) \leq f_y^1\left(y_{\frac{\vartheta q_0}{2}}^1 + a\right), \quad f_y^1\left(y_{1-\frac{\vartheta q_0}{2}}^1 - a\right) \geq f_y^1\left(y_{1-\frac{\vartheta q_0}{2}}^1 + a\right),$$

*and (ii) at the points  $y_{\vartheta q_0/2}^1$  and  $y_{1-\vartheta q_0/2}^1$ ,  $f_y^1$  is continuous and positive.*

Part (i) says that the shaded area in Figure 1b is below the conditional density  $f_y^1$ . Part (ii) is a weak assumption and is standard in the density estimation literature (e.g., Cattaneo et al., 2020). One can easily verify Assumption 3.2-(i) by drawing a density estimate using a standard estimation strategy. This visual inspection is useful to examine the potential for further narrowing the obtained bounds. Some discussion on this is provided in Section 5 using an empirical example.

Based on these assumptions, we obtain the following identification results:

THEOREM 3.1. *Suppose the model (2.1). Let  $Y_{0i}^*$  and  $Y_{1i}^*$  be continuous random variables. Assume Assumptions 2.1, 2.3, and 3.1 hold. Then,  $[\Delta^s, \nabla^s]$  are valid bounds, i.e.,  $\tau \in [\Delta^s, \nabla^s]$ , where*

$$\begin{aligned} \nabla^s &= y_{1-\frac{\vartheta q_0}{2}}^1 - \mathbb{E}[Y_i | D_i = 0, S_i = 1], \\ \Delta^s &= y_{\frac{\vartheta q_0}{2}}^1 - \mathbb{E}[Y_i | D_i = 0, S_i = 1], \end{aligned}$$

*and  $y_r^1$ ,  $q_0$ , and  $\vartheta$  are defined in Theorem 2.1. Besides, if Assumption 3.2 is also true, the lower and upper bounds,  $\Delta^s$  and  $\nabla^s$ , are attainable; that is, there exist some joint distributions of  $(Y_{1i}^*, Y_{0i}^*, S_{1i}, S_{0i}, D_i)$  that are consistent with the assumptions, induce distribution of  $(Y_i, S_i, D_i)$ , and satisfy  $\tau = \Delta^s$  or  $\tau = \nabla^s$ .*

REMARK 3.1. *The bounds obtained in the previous theorem may not be sharp, in the sense that some regions or points in  $(\Delta^s, \nabla^s)$  could be inconsistent with the data or assumptions, and the resulting sharp bounds may be a set of subintervals of  $[\Delta^s, \nabla^s]$ . Examples are provided in the Online Appendix. However, the endpoints of the bounds,  $\Delta^s$  and  $\nabla^s$ , cannot be rejected as the true value of  $\tau$ . Hence, the practical implication that “the true effect is greater than  $\Delta^s$  and less than  $\nabla^s$ , or equal to these values” remains intact, which is still useful. Of course, the bounds are guaranteed to be sharp with a certain additional condition, which is proved in the Online Appendix.*

Theorem 3.1 implies  $[\Delta^s, \nabla^s] \subseteq [\Delta, \nabla]$ , that is, we can obtain tightened bounds. Besides, as Theorem 3.1 suggests, the symmetry assumption can be used in conjunction with the stochastic monotonicity assumption, as illustrated by the following example:

EXAMPLE 3.1. (EXAMPLES 2.1 AND 2.4 CONTINUED) *Recall Examples 2.1 and 2.4, in which the monotonicity fails. The bounds under stochastic monotonicity with  $\vartheta_L = 0.95$  and symmetry are  $[-0.0026, 1.49]$ , which are valid and tighter than the bounds obtained in Example 2.4.*

#### 4. ESTIMATION AND INFERENCE

We consider the estimation and inference procedures. We focus on the lower bound obtained in Theorem 3.1 to avoid redundancy. The upper bound is analogous, and the procedures for the bounds obtained in Theorem 2.1 are similar, which are provided in the Online Appendix. We will consider two types of estimators in the subsequent subsections.

##### 4.1. When $\vartheta_L \geq \vartheta_F$ is Known

We first consider when  $\vartheta_L \geq \vartheta_F$  is known. The inequality  $\mathbb{P}[S_{1i} = 1 | S_{0i} = 1] \geq \vartheta_F$ , or equivalently  $\mathbb{P}[S_{1i} = 1 | S_{0i} = 0] \leq 1$ , is not a very strong requirement. Hence,  $\vartheta_F$  may not be very large. Thus,  $\vartheta_L \geq \vartheta_F$  can be satisfied in many applications, especially when  $\vartheta_L$  is large, e.g.,  $\vartheta_L = 0.95$ . Lee’s exact monotonicity ( $\vartheta_L = 1$ ) is an leading example of this case.

Define  $\beta := (\beta^L, q, \alpha, \eta)$  and  $\beta_0 := (\beta_0^L, q_0, \alpha_0, \eta_0)$ , where  $\beta_0^L = y_{\vartheta_L q_0/2}^1$  and  $\eta_0 = \mathbb{E}[Y_i | D_i = 0, S_i = 1]$ . Similarly to Lee (2009, p. 1099), defining

$$g(\beta) = \begin{pmatrix} (1\{Y_i > \beta^L\} - (1 - \vartheta_L q/2)) S_i D_i \\ (q S_i - \alpha) D_i \\ (S_i - \alpha)(1 - D_i) \\ (Y_i - \eta) S_i (1 - D_i) \end{pmatrix},$$

we can estimate  $\beta_0$  by  $\min_{\beta} (\sum_{i=1}^n g(\beta))' (\sum_{i=1}^n g(\beta))$ . Write the minimiser by  $\hat{\beta} = (\hat{\beta}^L, \hat{q}, \hat{\alpha}, \hat{\eta})$ . Then, we can estimate the lower bound by  $\hat{\Delta}_1^s = \hat{\beta}^L - \hat{\eta}$ . We have the next asymptotic normality.

COROLLARY 4.1. *Assume that  $Y_i$  have bounded support,  $f_y^1$  is continuous and  $0 < f_y^1 < \infty$ ,  $\mathbb{P}[S_i = 1 | D_i = 0] > 0$ ,  $\mathbb{P}[S_{1i} > S_{0i}] > 0$ , and Assumptions 2.1 and 2.3 hold. Then*

$\sqrt{n}(\widehat{\Delta}_1^s - \Delta^s) \rightarrow_d \mathcal{N}(0, \Omega_L + \Omega_C)$ , where

$$\begin{aligned}\Omega_L &= \left\{ f_y^1 \left( y_{\frac{\vartheta q_0}{2}}^1 \right)^2 \right\}^{-1} \frac{1}{\mathbb{P}[S_i = 1, D_i = 1]} \left\{ \frac{\vartheta q_0}{2} \left( 1 - \frac{\vartheta q_0}{2} \right) + \frac{\vartheta^2 q_0^2}{4} \Psi_1 + \frac{\vartheta^2}{4} \Psi_2 \right\}, \\ \Psi_1 &= \frac{\alpha_0}{q_0} \left( 1 - \frac{\alpha_0}{q_0} \right) \frac{\mathbb{P}[D_i = 1]}{\mathbb{P}[S_i = 1, D_i = 1]}, \quad \Psi_2 = \alpha_0(1 - \alpha_0) \left( \frac{\mathbb{P}[D_i = 1]}{\mathbb{P}[D_i = 0]} \right)^2 \frac{\mathbb{P}[D_i = 0]}{\mathbb{P}[S_i = 1, D_i = 1]}, \\ \text{and } \Omega_C &= \mathbb{V}[Y_i | S_i = 1, D_i = 0] / \mathbb{P}[S_i = 1, D_i = 0].\end{aligned}$$

We can define  $\widehat{\nabla}_1^s$  similarly and have  $\sqrt{n}(\widehat{\nabla}_1^s - \nabla^s) \rightarrow_d \mathcal{N}(0, \Omega_U + \Omega_C)$ , where

$$\Omega_U = \left\{ f_y^1 \left( y_{1 - \frac{\vartheta q_0}{2}}^1 \right)^2 \right\}^{-1} \frac{1}{\mathbb{P}[S_i = 1, D_i = 1]} \left\{ \frac{\vartheta q_0}{2} \left( 1 - \frac{\vartheta q_0}{2} \right) + \frac{\vartheta^2 q_0^2}{4} \Psi_1 + \frac{\vartheta^2}{4} \Psi_2 \right\}$$

The probabilities and conditional variance can be estimated by their sample analogues. The conditional density  $f_y^1$  can be consistently estimated by the standard kernel method, and standard bandwidth selectors can be used by assuming additional smoothness assumption on  $f_y^1$  (see Jones et al., 1996). Then, the asymptotic variances can be consistently estimated by the continuous mapping theorem. Therefore, similarly to Lee (2009), we can perform inference based on Imbens and Manski's (2004) confidence interval (CI).

#### 4.2. Unknown Case

Next, we consider the estimation and inference procedure when a researcher does not know which is larger,  $\vartheta_L$  and  $\vartheta_F$ . Define  $\gamma := (\gamma^L, \gamma^F, q, \alpha, \eta)$  and  $\gamma_0 := (\gamma_0^L, \gamma_0^F, q_0, \alpha_0, \eta_0)$ , where  $\gamma_0^L = y_{\vartheta_L q_0/2}^1$  and  $\gamma_0^F = y_{(1+q_0(1-1/\alpha_0))/2}^1$ . Defining

$$\tilde{g}(\gamma) = \begin{pmatrix} (\mathbf{1}\{Y_i > \gamma^L\} - (1 - \vartheta_L q/2)) S_i D_i \\ (\mathbf{1}\{Y_i > \gamma^F\} - (1 - (1 + q(1 - 1/\alpha))/2)) S_i D_i \\ (q S_i - \alpha) D_i \\ (S_i - \alpha)(1 - D_i) \\ (Y_i - \eta) S_i (1 - D_i) \end{pmatrix},$$

we can estimate  $\gamma_0$  by  $\min_{\gamma} (\sum_{i=1}^n \tilde{g}(\gamma))' (\sum_{i=1}^n \tilde{g}(\gamma))$ . We represent the minimiser by  $\hat{\gamma} = (\hat{\gamma}^L, \hat{\gamma}^F, \hat{q}, \hat{\alpha}, \hat{\eta})$ . We can estimate  $\Delta^s$  by the following:

$$\widehat{\Delta}_2^s = \max_{v \in \{L, F\}} (\hat{\gamma}^v - \hat{\eta}).$$

Now, we have the following lemma:

LEMMA 4.1. *Under the same assumptions in Corollary 4.1,  $\sqrt{n}(\hat{\gamma} - \gamma_0) \rightarrow_d \mathcal{N}(0, \Omega_{\gamma, L})$ , where  $\Omega_{\gamma, L}$  is given in the Online Appendix.*

Then, we can apply the procedure developed by Chernozhukov et al. (2013) for inference. Conceptually, Chernozhukov et al.'s (2013) CI is different from Imbens and Manski's (2004) in that the former is the CI for the identified region, i.e., the bounds  $[\Delta^s, \nabla^s]$ , while the latter is for the true parameter,  $\tau$ . Therefore, the CI based on Chernozhukov et al. (2013) can be conservative for  $\tau$ , although valid.

## 5. EMPIRICAL ILLUSTRATION

## 5.1. Empirical Contexts and Main Results

We illustrate the usefulness of the bounds based on the stochastic monotonicity and symmetry assumptions using the data from Muralidharan et al. (2019, 2022). The authors used a randomised controlled trial to examine the effect of the after-school education programme, called Mindspark intervention, on math and Hindi test scores.

The Mindspark intervention consists of two sessions, technology-led and group-based instructions, while the instruction was mainly provided in the former session (Muralidharan et al., 2019, p. 1427). This technology-led instruction is a computer-based interactive instruction. The content provided in this session is automatically personalised for each student, and this adaptation is performed dynamically based on the beginning assessment and every subsequent activity completed.

The educational attainment is assessed by the baseline and endline tests. These tests are designed to capture a broad range of students' achievements. The test questions ranged in difficulty from "very easy" to "grade-appropriate" levels (Muralidharan et al., 2019, p. 1436).

In this intervention, the lottery winners (314 students,  $D_i = 1$ ) were assigned to the treatment group, and the losers (305 students,  $D_i = 0$ ) were assigned to the control group.  $S_{di}$  denotes whether student  $i$  takes the endline test when  $D_i = d$ . We define the outcome in interest ( $Y_{1i}^*$  and  $Y_{0i}^*$ ) as the difference between the endline and baseline test scores.<sup>4</sup>

One problem with the Mindspark intervention was attrition. 15.3% and 10.5% dropped out from the treatment and control groups, respectively. To address the potential endogenous sample selection, Muralidharan et al. (2019) computes the Lee bounds. They assume that the exact monotonicity in the selection, or  $S_{0i} \geq S_{1i}$  holds almost surely. Note that the direction of the monotonicity is opposite to what was assumed in the previous sections. An implicit assumption might be that untreated students were more likely to take the endline test because they "would be provided free access" (Muralidharan et al., 2019, p. 1434) to the program after the experiment if they participated in the endline test, whereas such free access was not offered to the treated group. This could have incentivised those with  $D_i = 0$  not to attrit more than those with  $D_i = 1$ .

We first compute the Lee bounds based on this Muralidharan et al.'s (2019) assumption (i.e.,  $\mathbb{P}[S_{0i} = 1 | S_{1i} = 1] = 1$ ). The estimated bounds and Imbens and Manski's (2004) CIs are reported in rows (1) and (3) of Table 1. The CI for the math scores is (0.153, 0.627), and thus, the difference in test scores between the endline and baseline for always-takers is positive. In contrast, the CI for the Hindi scores is (−0.018, 0.423) and covers zero. This may imply that the possibility of no effect cannot be rejected.

However, this may perhaps be due to the "extreme" trimming of the Lee bounds. To address this point, we estimate bounds under the symmetry assumption, whose validity is discussed in the subsequent subsection. Now,  $1 = \vartheta_L \geq \vartheta_F$  is satisfied, and thus, we can perform the estimation/inference procedure outlined in Section 4.1. The estimation

<sup>4</sup>Muralidharan et al. (2019, Table A8) uses a different indicator when computing the Lee bounds. They use "residuals from a regression of endline test scores on baseline test scores (value-added)" using the whole sample (except for attriters) as the outcome. However, this approach seems to fail to account for a likely difference in average improvements in test scores between the treatment and control groups. Therefore, this regression residual may have limitations as an indicator of students' achievement. Here, we simply use the difference in individual test scores, as it is both straightforward and interpretable.

**Table 1:** Empirical Bounds under Monotonicity and Symmetry

|          | <b>Math</b>    |                | <b>Hindi</b>    |                |
|----------|----------------|----------------|-----------------|----------------|
|          | (1)            | (2)            | (3)             | (4)            |
| Symmetry |                | Yes            |                 | Yes            |
| Bounds   | [0.291, 0.494] | [0.313, 0.424] | [0.110, 0.293]  | [0.118, 0.172] |
| CI       | (0.153, 0.627) | (0.182, 0.558) | (−0.018, 0.423) | (0.007, 0.286) |

**Table 2:** Empirical Bounds under Stochastic Monotonicity and Symmetry

|          | <b>Math</b>     |                 |                 |
|----------|-----------------|-----------------|-----------------|
|          | (1)             | (2)             | (3)             |
| Symmetry |                 | Yes             | Yes             |
| Bounds   | [0.227, 0.556]  | [0.282, 0.479]  | [0.282, 0.479]  |
| CI       | (0.095, 0.684)  | (0.153, 0.612)  | (0.106, 0.658)  |
|          | <b>Hindi</b>    |                 |                 |
|          | (4)             | (5)             | (6)             |
| Symmetry |                 | Yes             | Yes             |
| Bounds   | [0.048, 0.357]  | [0.099, 0.186]  | [0.099, 0.186]  |
| CI       | (−0.075, 0.476) | (−0.008, 0.298) | (−0.045, 0.334) |

**Note:**  $\vartheta_L = 0.95$ . Cases (1), (2), (4), and (5) are based on Section 4.1 and Imbens and Manski (2004). Cases (3) and (6) are based on Section 4.2 and Chernozhukov et al. (2013).

results are shown in rows (2) and (4) of Table 1. The obtained bounds are tighter than the Lee bounds. The lengths of the bounds for math and Hindi scores are around 46% and 70% shorter, respectively. The CIs are also tightened; their lengths are approximately 21% and 37% shorter than those of the Lee bounds. Besides, the CI for the Hindi scores does not cover zero, suggesting a positive treatment effect under monotonicity and symmetry. These results showcase the possible usefulness of the new bounds obtained in Theorem 3.1.

So far, we have considered bounds when the monotonicity exactly holds. However, a possible concern is the validity of this monotonicity assumption. Recall that Muralidharan et al. (2019) assumes that if a treated student took the end-line test, they must have taken the test if they were in the control group. This assumption might be justified because the untreated students “were told that they would be provided free access” to the programme after the end of the experiment if they took the end-line test (Muralidharan et al., 2019, p. 1434). However, it would not be surprising if some students with  $S_{1i} = 1$  dropped out when not in the treatment group: For example, students who did not win the lottery might have sought alternative educational resources (e.g., textbooks, other tutoring services) and then might no longer be interested in participating in the Mindspark programme by the time of the endline test. Alternatively, not participating in the Mindspark intervention could cause parents to overlook the endline test, potentially leading to scheduling conflicts (e.g., travel, shopping, assisting with parents’ work) on the test day, thereby preventing their children from attending.

Hence, we compute the bounds under stochastic monotonicity. Considering the incentive provided to the lottery losers, we assume  $\vartheta_L = 0.95$ . The results are summarised in Table 2.

The results for the math scores are reported in rows (1)–(3). The treatment effect is still positive even if the monotonicity assumption does not exactly hold, which suggests the robustness of the positive effect of the Mindspark intervention on the math scores. The results for Hindi are in contrast. Although the bounds and CIs in rows (5)–(6) are much tighter than those without symmetry in row (4), the CIs for the Hindi scores under stochastic monotonicity cover zero. This means that the treatment effect can be zero if we allow the monotonicity to be only slightly violated. Therefore, the validity of the exact monotonicity will be important to assess whether the Mindspark intervention improved the students’ Hindi scores. This illustrates the importance of relaxing monotonicity and examining the sensitivity of the results under the exact monotonicity. The stochastic monotonicity assumption is useful for such an objective.

### 5.2. Validity of Assumptions

Below, we provide an illustrative discussion of the validity of our assumptions.

*Mean-Median Coincidence* In the context of the Mindspark intervention, the mean-median coincidence assumption could be reasonable. As noted before, the main instruction provided via Mindspark software is automatically personalised. Besides, the tests include a wide range of questions from easy to difficult, designed to measure the various improvements of students. Considering these points, we may assume that the improvement in test scores is similar across students, and the observed differences may stem from stochastic reasons, such as compatibility with the questions. Then, we may assume  $Y_{1i}^* = t + \varepsilon_i$ , where  $t \in \mathbb{R}$  and  $\mathbb{E}[\varepsilon_i] = \text{Median}[\varepsilon_i] = 0$  such as the Gaussian error. This certain homogeneity assumption is consistent with Muralidharan et al. (2019, Section III.C), wherein they found limited evidence of heterogeneity in students’ progress by their initial learning level, gender, and socioeconomic status.

*Tail Smoothness* We assess whether the lower bound on Hindi scores under symmetry (Table 2 (5)–(6)) is attainable or not by visually verifying Assumption 3.2-(i). Let  $F_0$  and  $f_y^0$  be the distribution and density functions, respectively, of  $Y_i$  conditional on  $D_i = 0$  and  $S_i = 1$ ; and  $y_r^0 = F_0^{-1}(r)$ . In Figure 2, we show the kernel density estimates of  $f_y^0$ , 99% point-wise robust bias-corrected (RBC) CIs of Calonico et al. (2018), and their folded ones at  $\hat{y}_{1-\hat{\vartheta}\hat{q}_0/2}^0$ , which is indicated by a dotted vertical line. We find no strong evidence suggesting Assumption 3.2-(i) is refuted; the folded line is always below the original density estimates, and CIs do not indicate that the former is above the latter. Hence, the lower bound is not rejected.

*Stochastic Monotonicity* When the upper bound in (2.4) is too small, it may suggest that Assumption 2.3-(i) could be less plausible. As the direction of the monotonicity is opposite in our example, we compute the empirical upper bound on  $\mathbb{P}[S_{0i} = 1 | S_{1i} = 0]$ , finding it to be 0.56, which does not appear too small. Assumption 2.3-(ii) may be justified by the premise that most of the students with  $S_{1i} = 1$  are motivated and would likely take the endline test to participate in the program even if they lost the lottery, and that the attrition resulting from the previously mentioned taste shock or overlapping schedules can be regarded as a random shock.

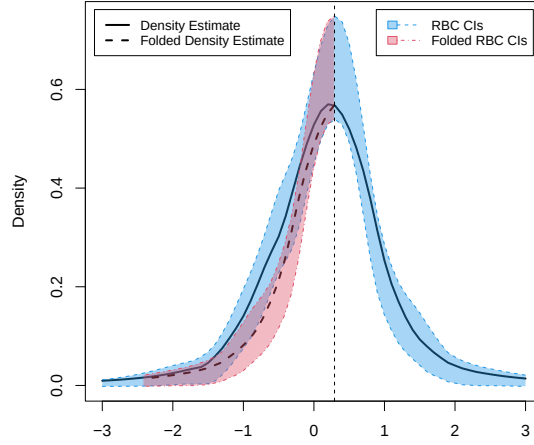


**Figure 2:** Density Estimate of  $f_y^0$  (Hindi)

## 6. CONCLUSION

This study considered the partial identification of Lee’s general sample selection model under a stochastic version of the monotonicity and symmetry assumptions. The former assumption was introduced to robustify the Lee bounds to the small deviation from the exact monotonicity, while the latter assumption can be used to tighten the Lee bounds, which may be empirically wide. The obtained bounds are root- $n$  consistently estimable, making it easy to perform estimation and inference in practice.

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#### APPENDIX: PROOF OF THEOREMS

In the proof of Theorem 2.1, we utilise Proposition 4 of Horowitz and Manski (1995), which we reproduce below. Let  $F$  and  $G$  be distributions and  $F$  (first-order) stochastically dominates  $G$ . We say that a real-valued functional  $\rho$  respects stochastic dominance if  $\rho(F) \geq \rho(G)$ .

**Proposition 4 of Horowitz and Manski (1995):** Let  $Y$  be a continuous random variable from a distribution  $Q$  such that  $Q = pP_0 + (1 - p)P_1$ , and  $y_q$  be the  $q$ -th quantile

of  $Q$ . Assume we know that  $p \leq \lambda < 1$  and  $\rho$  respects stochastic dominance. Define the distributions

$$\begin{aligned} L_\lambda(t) &= Q(t)/(1-\lambda) \times \mathbf{1}\{t < y_{1-\lambda}\} + \mathbf{1}\{t \geq y_{1-\lambda}\}, \\ U_\lambda(t) &= (Q(t) - \lambda)/(1-\lambda) \times \mathbf{1}\{t \geq y_\lambda\}. \end{aligned}$$

Then,  $\rho(P_1) \in [\rho(L_\lambda), \rho(U_\lambda)]$  and the bounds are sharp.

**Proof of Theorem 2.1:** Note that

$$\mathbb{E}[Y_{0i}^* | S_{0i} = S_{1i} = 1] = \mathbb{E}[Y_{0i}^* | S_{0i} = 1] = \mathbb{E}[Y_i | D_i = 0, S_i = 1],$$

where the first equality follows from Assumption 2.3-(ii) and the second from Assumption 2.1. Thus, it suffices to show the first terms of  $\nabla$  and  $\Delta$  are sharp bounds. Below, we assume  $\vartheta_L < 1$  as the case  $\vartheta_L = 1$  reduces to the Lee's (2009) result. Observe

$$\begin{aligned} \mathbb{P}[Y_i \leq y | D_i = 1, S_i = 1] \\ = \mathbb{P}[Y_{1i}^* \leq y | S_{0i} = 0, S_{1i} = 1] p_0 + \mathbb{P}[Y_{1i}^* \leq y | S_{0i} = 1, S_{1i} = 1] (1 - p_0), \end{aligned} \quad (\text{A.1})$$

where  $p_0 = \mathbb{P}[S_{0i} = 0, S_{1i} = 1 | D_i = 1] / \mathbb{P}[S_i = 1 | D_i = 1]$ . By Assumption 2.1 and Assumption 2.3,  $1 - p_0$  can be bounded from below as

$$\begin{aligned} 1 - p_0 &= \frac{\mathbb{P}[S_{0i} = 1, S_{1i} = 1 | D_i = 1]}{\mathbb{P}[S_i = 1 | D_i = 1]} = \frac{\mathbb{P}[S_{0i} = 1, S_{1i} = 1 | D_i = 0]}{\mathbb{P}[S_i = 1 | D_i = 1]} \\ &= \frac{\mathbb{P}[S_i = 1 | D_i = 0] \mathbb{P}[S_{1i} = 1 | S_{0i} = 1]}{\mathbb{P}[S_i = 1 | D_i = 1]} \geq \vartheta_L q_0, \end{aligned}$$

or equivalently,  $p_0 \leq 1 - \vartheta_L q_0$ . Plus, applying the Fréchet inequality to  $\mathbb{P}[S_{1i} = 1 | S_{0i} = 1]$ , we have  $1 - p_0 \geq \vartheta_F q_0$ , i.e.,  $p_0 \leq 1 - \vartheta_F q_0$ . Hence,  $p_0 \leq 1 - \vartheta q_0$  with  $\vartheta = \max\{\vartheta_L, \vartheta_F\}$ . Note that the expectation operator respects stochastic dominance. Then, applying Proposition 4 of Horowitz and Manski (1995) to (A.1), we find that the first terms of  $\nabla$  and  $\Delta$  are the sharp bounds on  $\mathbb{E}[Y_{1i}^* | S_{0i} = S_{1i} = 1]$ .  $\square$

**Proof of Theorem 3.1:** We focus on the upper bound as the proof for the lower bound is similar. Firstly, we show the validity of  $\nabla^s$ , that is,

$$\mathbb{E}[Y_{1i}^* | S_{0i} = S_{1i} = 1] \leq y_{1-\frac{\vartheta q_0}{2}}^1.$$

To show this, assume  $\mathbb{E}[Y_{1i}^* | S_{0i} = S_{1i} = 1] > y_{1-\frac{\vartheta q_0}{2}}^1$ . Recalling (A.1), it follows that

$$\mathbb{P}[Y_i > y_{1-\frac{\vartheta q_0}{2}}^1 | D_i = 1, S_i = 1] \geq \mathbb{P}[Y_{1i}^* > y_{1-\frac{\vartheta q_0}{2}}^1 | S_{0i} = 1, S_{1i} = 1] (1 - p_0).$$

Then, we can compute that

$$\begin{aligned} \frac{\vartheta q_0}{2} &\geq \mathbb{P}[Y_i > y_{1-\frac{\vartheta q_0}{2}}^1 | D_i = 1, S_i = 1] \\ &\geq \mathbb{P}[Y_{1i}^* > y_{1-\frac{\vartheta q_0}{2}}^1 | S_{0i} = 1, S_{1i} = 1] (1 - p_0) \\ &> \frac{1}{2} \times (1 - p_0) \\ &\geq \frac{\vartheta q_0}{2}, \end{aligned} \quad (\text{A.2})$$

where the first inequality uses the definition of  $y_r^1$ , and (A.2) holds under Assumption 3.1

and  $\mathbb{E}[Y_{1i}^* | S_{0i} = S_{1i} = 1] > y_{1-\vartheta q_0/2}^1$ . Thus, contradiction arises and  $\mathbb{E}[Y_{1i}^* | S_{0i} = S_{1i} = 1] \leq y_{1-\vartheta q_0/2}^1$  is proved.

Next, we additionally assume Assumption 3.2. We will show that the upper bound  $\nabla^s$  is attainable. We construct a joint distribution of  $(\tilde{Y}_{1i}^*, \tilde{Y}_{0i}^*, \tilde{S}_{1i}, \tilde{S}_{0i}, D_i)$  that is consistent with the assumptions, attains the upper bound, and induces the joint distribution of  $(Y_i, S_i, D_i)$ . Firstly, define  $\mathbb{P}[\tilde{S}_{0i} = 1] := \mathbb{P}[S_i = 1 | D_i = 0]$ ,  $\mathbb{P}[\tilde{S}_{1i} = 1] := \mathbb{P}[S_i = 1 | D_i = 1]$ ,  $\mathbb{P}[\tilde{Y}_{0i}^* \leq y | \tilde{S}_{0i} = 1] := \mathbb{P}[Y_i \leq y | D_i = 0, S_i = 1]$ ,  $\mathbb{P}[\tilde{Y}_{1i}^* \leq y | \tilde{S}_{1i} = 1] := \mathbb{P}[Y_i \leq y | D_i = 1, S_i = 1]$ , and  $\mathbb{P}[\tilde{S}_{1i} = 1 | \tilde{S}_{0i} = 1] := \vartheta$ . Then, we can obtain that

$$\begin{aligned}\pi_{11} &:= \mathbb{P}[\tilde{S}_{0i} = 1, \tilde{S}_{1i} = 1] = \vartheta \mathbb{P}[S_i = 1 | D_i = 0], \\ \pi_{10} &:= \mathbb{P}[\tilde{S}_{0i} = 1, \tilde{S}_{1i} = 0] = (1 - \vartheta) \mathbb{P}[S_i = 1 | D_i = 0], \\ \pi_{01} &:= \mathbb{P}[\tilde{S}_{0i} = 0, \tilde{S}_{1i} = 1] = \mathbb{P}[S_i = 1 | D_i = 1] - \vartheta \mathbb{P}[S_i = 1 | D_i = 0], \\ \pi_{00} &:= \mathbb{P}[\tilde{S}_{0i} = 0, \tilde{S}_{1i} = 0] = 1 - \pi_{11} - \pi_{10} - \pi_{01},\end{aligned}$$

and  $\mathbb{P}[\tilde{Y}_{di}^* \leq y, \tilde{S}_{di} = 1] = \mathbb{P}[Y_i \leq y, S_i = 1 | D_i = d]$ ,  $d \in \{0, 1\}$ . Next, we define

$$\begin{aligned}\pi_{1|1}^* &:= \mathbb{P}[\tilde{Y}_{1i}^* \leq y | \tilde{S}_{0i} = 1, \tilde{S}_{1i} = 1] \\ &:= \begin{cases} \frac{1}{2} + \frac{1}{\vartheta q_0} \mathbb{P}\left[y_{1-\frac{\vartheta q_0}{2}}^1 \leq \tilde{Y}_{1i}^* \leq y | \tilde{S}_{1i} = 1\right] & \text{if } y \geq y_{1-\frac{\vartheta q_0}{2}}^1, \\ \frac{1}{\vartheta q_0} \mathbb{P}\left[2y_{1-\frac{\vartheta q_0}{2}}^1 - y \leq \tilde{Y}_{1i}^* | \tilde{S}_{1i} = 1\right] & \text{if } y < y_{1-\frac{\vartheta q_0}{2}}^1, \end{cases} \\ \mathbb{P}[\tilde{Y}_1^* \leq y | \tilde{S}_{0i} = 1, \tilde{S}_{1i} = 0] &:= \pi_{1|1}^*, \\ \mathbb{P}[\tilde{Y}_1^* \leq y | \tilde{S}_{0i} = 0, \tilde{S}_{1i} = 1] &:= \frac{1}{1 - \vartheta q_0} \mathbb{P}[\tilde{Y}_{1i}^* \leq y | \tilde{S}_{1i} = 1] - \frac{\vartheta q_0}{1 - \vartheta q_0} \pi_{1|1}^*, \quad (\text{A.3}) \\ \mathbb{P}[\tilde{Y}_1^* \leq y | \tilde{S}_{0i} = 0, \tilde{S}_{1i} = 0] &:= \mathbb{P}[\tilde{Y}_1^* \leq y | \tilde{S}_{1i} = 1]\end{aligned}$$

and  $\mathbb{P}[\tilde{Y}_0^* \leq y | \tilde{S}_{0i} = s_0, \tilde{S}_{1i} = s_1] := \mathbb{P}[\tilde{Y}_0^* \leq y | \tilde{S}_{0i} = 1]$  for all  $s_0, s_1 \in \{0, 1\}$ . Assume  $\tilde{Y}_{1i}^* \perp\!\!\!\perp \tilde{Y}_{0i}^* | (\tilde{S}_{1i}, \tilde{S}_{0i})$ . Then Assumptions 2.3 and 3.1 are satisfied under Assumption 3.2-(ii). Also, (A.3) is a proper distribution function under Assumption 3.2-(i). We can write the joint distribution as a product of the defined conditional probabilities under Assumption 2.1. Besides, letting  $\tilde{S}_i = \tilde{S}_{1i} D_i + \tilde{S}_{0i} (1 - D_i)$ ,  $\tilde{Y}_i = \tilde{S}_i \cdot \left\{ \tilde{Y}_{1i}^* D_i + \tilde{Y}_{0i}^* (1 - D_i) \right\}$ , it follows that  $\mathbb{P}[\tilde{Y}_i \leq y, \tilde{S}_i = 1, D_i = 1] = \mathbb{P}[\tilde{Y}_{1i}^* \leq y, \tilde{S}_{1i} = 1] \mathbb{P}[D_i = 1] = \mathbb{P}[Y_i \leq y, S_i = 1, D_i = 1]$ , and that  $\mathbb{P}[\tilde{Y}_i \leq y, \tilde{S}_i = 1, D_i = 0] = \mathbb{P}[Y_i \leq y, S_i = 1, D_i = 0]$  similarly. Hence, it induces the joint distribution of the data  $(Y_i, S_i, D_i)$ . Finally,

$$\nabla^s = \mathbb{E}[\tilde{Y}_1^* | \tilde{S}_{0i} = 1, \tilde{S}_{1i} = 1] - \mathbb{E}[\tilde{Y}_0^* | \tilde{S}_{0i} = 1] = \mathbb{E}[\tilde{Y}_1^* - \tilde{Y}_0^* | \tilde{S}_{0i} = 1, \tilde{S}_{1i} = 1],$$

where the first equality follows by construction. Hence, the upper bound is attainable.  $\square$